

DETERMINATION OF THE ABSTRACT GROUPS OF ORDER p^2qr ;

p, q, r BEING DISTINCT PRIMES*

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Since the publication† in 1899 of Professor MILLER's "Report on recent progress in the theory of groups of finite order," WESTERN‡ has published his determination of the groups of order p^3q , and LE VASSEUR§ has discussed the order p^2q^2 . This paper is devoted to the determination of all groups of the order p^2qr . It thus completes the discussion of the problem of groups whose orders are products of four primes. ||

With the exception of the group of order $2^2 \cdot 3 \cdot 5$, simply isomorphic with the icosahedron-group, all groups of order p^2qr are solvable. The maximal self-conjugate subgroups will therefore serve as the basis of classification. The twelve possible arrangements of the factors of composition are

- (1) $ppqr$, (2) $pprq$, (3) $pqpr$, (4) $pqrp$, (5) $prpq$, (6) $prqp$,
(7) $qppr$, (8) $qppr$, (9) $qrpp$, (10) $rqp p$, (11) $rppq$, (12) $rpqp$.

If for a given type of group precisely the arrangements $(i), (j), (k), \dots$, of the factors of composition are possible, then we symbolize¶ the group $(i, j, k, \dots)$. Two groups having distinct symbols cannot be simply isomorphic.

The group G always contains a maximal invariant subgroup** of order p^2q , and may contain maximal subgroups†† of order p^2r and pqr . We shall discuss

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† Bulletin, American Mathematical Society, vol. 1 (1899), p. 227.

‡ Proceedings of the London Mathematical Society, vol. 30 (1899), p. 209.

§ Annales Toulouse, 1903, p. 63. Comptes Rendus, vol. 128 (1899), p. 1152, and lithographed book.

|| HÖLDER, Mathematische Annalen, vol. 43 (1893), p. 335. BURNSIDE, *Finite Groups*, p. 81. HÖLDER, Göttinger Nachrichten (1895), p. 211.

¶ Additional abbreviations used throughout are the following: $P, Q, \dots$, operations of order $p, q, \dots$; $H_{\lambda, i}$, a maximal invariant subgroup of G , order h and type i ; $\rho_{\Omega, \lambda}$, number of subgroups of G , order h , permutable with Ω ; N_λ , number of subgroups of G of order h .

** FROBENIUS, Berliner Sitzungsberichte, vol. 1 (1895), p. 170.

†† HÖLDER, loc. cit., COLE and GLOVER, American Journal of Mathematics, vol. 15 (1893), p. 202. BURNSIDE, *Theory of Groups*, p. 63.

in detail in this paper only two classes of groups: those possessing invariant subgroups of both the types H_{p^2q} and H_{p^2r} , and those possessing maximal invariant subgroups of the type H_{p^2q} only. A detailed summary of the results obtained in the other classes is given at the end. We shall thus be concerned principally with the subgroups $H_{p^2\sigma}$ ($\sigma = q, r$) all types of which are given in the following table, in which τ denotes the number of distinct types, while (p) signifies (modulo p):

$H_{p^2\sigma}, i$	$S_3^{-1}SS_3$	$S_3^{-1}S_1S_3$	$S_3^{-1}S_2S_3$	$S_2^{-1}S_1S_2$	Parameters	τ
$i = I$	S	.	.	.		1
II	.	S_1	S_2	S_1		1
III	.	S_1^a	S_2	S_1	$\alpha^\sigma \equiv 1(p), p \equiv 1(\sigma)$	1
IV	S^a	.	.	.	$\alpha^\sigma \equiv 1(p^2), p \equiv 1(\sigma)$	1
V	.	S_1^a	S_2^a	S_1	$\alpha^\sigma \equiv 1(p), p \equiv 1(\sigma) \text{ 1 or } \frac{1}{2}[\sigma+1]$	
VI	.	S_2	$S_1^{-1}S_2^{p+1}$	S_1	$* \alpha^\sigma \equiv 1(p), p \equiv -1(\sigma)$	1

$\sigma = q, r; S^{p^2} = 1, S_1^p = S_2^p = 1, S_3^p = 1.$

§ 1. Determination of $\rho_{\Omega, \lambda}$.

By SYLOW's theorem,† $N_\sigma = qr/\sigma, p, p^2, pqr/\sigma, p^2qr/\sigma$ or 1. If $N_{\sigma, \lambda} = 1$ then $\rho_{\Omega, \sigma} = 1$, Ω being any operator of prime order in G . When $N_\sigma > 1$, the result of transforming the single conjugate set of N_σ subgroups

$$g_1, g_2, g_3, \dots, g_{N_\sigma}$$

by Ω is to permute them among themselves. Hence

$$\Omega^{-1}(g_1, g_2, \dots, g_{N_\sigma})\Omega = \begin{pmatrix} g_1, g_2, \dots, g_{N_\sigma} \\ g_{i_1}, g_{i_2}, \dots, g_{i_{N_\sigma}} \end{pmatrix} = J_{\Omega, \sigma}.$$

It follows that $J_{\Omega, \sigma}^\omega = 1$ and

$$(1) \quad N_\sigma - \rho_{\Omega, \sigma} \equiv 0 \pmod{\omega}; \quad \rho_{\Omega, \sigma} \geq 1.$$

Next let $\omega = \sigma$. Then $N_p = (p^2 - 1)/(p - 1) = p + 1$, and

$$(2) \quad p + 1 - \rho_{\Omega, p} \equiv 0 \pmod{\sigma}.$$

Hence either $\rho_{\Omega, p} = 0$ or else $\rho_{\Omega, p} \geq 2$ ($\omega = q, r$). Now if the subgroup I_{p^2} of $H_{p^2\sigma, \lambda}$ is cyclical the order of its group of isomorphisms is

$$I = \phi(p^2) = p(p - 1).$$

* Throughout the paper α denotes a non-integral mark of the $GF[p^2]$. Thus $\alpha^\sigma \equiv 1(p)$ is an abbreviation for $\alpha^\sigma \equiv 1 \pmod{p, P}$, P being any quadratic function irreducible modulo p .

† SYLOW, *Mathematische Annalen*, vol. 5 (1872).

If I_{p^2} is of type $[1, 1]$ its group of isomorphisms is simply isomorphic with the congruence group $\{S_1, S_2, \dots\}$ of order $I = p(p-1)^2(p+1)$, where S_1 is

$$y_1 \equiv a_{11}x_1 + a_{12}x_2, \quad y_2 \equiv a_{21}x_1 + a_{22}x_2 \pmod{p},$$

or say

$$S_1 = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2).$$

Since Ω corresponds to an isomorphism of G , $\{\Omega\}$ corresponds to a subgroup of the group of isomorphisms of G and ω divides I . Hence when I_{p^2} is cyclical, or when $I_{p^2} = [1, 1]$ and $p \equiv 1(\sigma)$, $\rho_{\Omega, p} \geq 2$. But when $p \equiv -1(\sigma)$ and p is odd, $\rho_{\Omega, p} = 0$. Also since $\rho_{\Omega, \sigma} \geq 1$, $J_{Q, \sigma}$ and $J_{R, \sigma}$ may be permutable. If

$$S_2 = (b_{11}x_1 + b_{12}x_2, b_{21}x_1 + b_{22}x_2)$$

the necessary and sufficient conditions that $S_1 S_2 = S_2 S_1$ are

$$(3) \quad \delta_{12} = \begin{vmatrix} a_{12} & b_{12} \\ a_{11} - a_{22} & b_{11} - b_{22} \end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix} a_{21} & b_{21} \\ a_{11} - a_{22} & b_{11} - b_{22} \end{vmatrix} \equiv 0,$$

$$d_{12} = \begin{vmatrix} a_{12} & a_{21} \\ b_{12} & b_{21} \end{vmatrix} \equiv 0.$$

§ 2. Class (9, 10), $p > q > r$.

We now consider the groups whose symbol is (9, 10), having the maximal subgroups $H_{p^2q, i}$ and $H_{p^2r, j}$ ($i, j = \text{IV, V, VI}$). Since I_{p^2} is invariant in G the existence of a subgroup of type IV excludes the possibility of a subgroup of type V or VI, and vice versa. There are thus five cases to consider.

[1] $i = j = \text{IV}$. Here $I_{p^2} = \{P\}$ is cyclical and P may be regarded as the generator of order p^2 in both H -sub-groups. Since $\rho_{Q, r} \geq 1$, we may choose $\{R\}$ permutable with Q and, since $q > r$, $QR = RQ$, so that G is defined by $P^{p^2} = Q^q = R^r = 1$, $Q^{-1}PQ = P^a$, $R^{-1}PR = P^b$, $QR = RQ$;

or for brevity $G = (\alpha : \beta : 1)$, where

$$\alpha \equiv 1, \quad \beta^r \equiv 1(p^2), \quad p \equiv 1(qr), \quad \tau = 1.$$

[2] $i = j = \text{V}$. Let $H_{p^2q, i} = \{P'_1, P'_2, Q\}$, $H_{p^2r, j} = \{P_1, P_2, R\}$, wherein $QR = RQ$. We may write

$$R^{-1}P_1R = P_1^\alpha, \quad R^{-1}P_2R = P_2^\beta, \quad \alpha^r \equiv 1(p), \quad \beta \equiv \alpha^t.$$

$$Q^{-1}P_1Q = P_1^{a_{11}}P_2^{a_{21}}, \quad Q^{-1}P_2Q = P_1^{a_{12}}P_2^{a_{22}},$$

and from the permutable isomorphisms of I_{p^2}

$$J_Q = \begin{pmatrix} P_1^{x_1} P_2^{x_2} \\ P_1^{a_{11}x_1 + a_{12}x_2} P_2^{a_{21}x_1 + a_{22}x_2} \end{pmatrix}, \quad J_R = \begin{pmatrix} P_1^{x_1} P_2^{x_2} \\ P_1^{ax_1} P_2^{\beta x_2} \end{pmatrix},$$

* All congruences are taken modulo v unless otherwise indicated

$$(4) \quad \delta_{12} = a_{12}(\alpha - \beta) \equiv 0, \quad \delta'_{12} = a_{21}(\alpha - \beta) \equiv 0.$$

Reserving for later treatment the ambiguous case $h = 1$, we deduce $a_{12} \equiv a_{21} \equiv 0$. Suppose next that

$$R^{-1}P'_iR = P_1^{b_{1i}}P_2^{b_{2i}} \quad (i = 1, 2).$$

Then

$$(5) \quad \begin{aligned} (RQ)^{-1}P'_1(RQ) &= P_1^{a_{11}b_{11}}P_2^{a_{22}b_{21}} = (QR)^{-1}P'_1(QR) = P_1^{\gamma b_{11}}P_2^{\gamma b_{21}}, \\ b_{11}(a_{11} - \gamma) &\equiv 0, \quad b_{21}(a_{22} - \gamma) \equiv 0, \quad \gamma^2 \equiv 1, \\ b_{12}(a_{11} - \delta) &\equiv 0, \quad b_{22}(a_{22} - \delta) \equiv 0, \quad \delta \equiv \gamma^k. \end{aligned}$$

Thus when $h \neq 1$, $k \neq 1$ we have one of the two equivalent results

$$a_{11} \equiv \gamma, \quad a_{22} \equiv \delta \quad \text{or} \quad a_{11} \equiv \delta, \quad a_{22} \equiv \gamma.$$

In case $h \neq 1$, $k = 1$, the set (5) becomes

$$\begin{aligned} b_{11}(a_{11} - \gamma) &\equiv 0, \quad b_{21}(a_{22} - \gamma) \equiv 0, \\ b_{12}(a_{11} - \gamma) &\equiv 0, \quad b_{22}(a_{22} - \gamma) \equiv 0, \end{aligned}$$

and there are three possibilities to consider, viz.,

- (i) $a_{11} \not\equiv \gamma, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0, \quad b_{21} \not\equiv 0, \quad b_{22} \not\equiv 0, \quad a_{22} \equiv \gamma;$
- (ii) $a_{11} \equiv \gamma, \quad a_{22} \not\equiv \gamma, \quad b_{21} \equiv b_{22} \equiv 0, \quad b_{11} \not\equiv 0, \quad b_{12} \not\equiv 0;$
- (iii) $a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma.$

Case (i) implies

$$\begin{aligned} R^{-1}P'_1R &= P_2^{b_{21}}, \quad R^{-1}P'_2R = P_2^{b_{22}}, \\ R^{-1}P_1^{b_{22}}R &= R^{-1}P_2^{b_{21}}R \quad \text{or} \quad P_1^{b_{22}} = P_2^{b_{21}}, \end{aligned}$$

contrary to the independence of P'_1 and P'_2 . Likewise, case (ii) is excluded. Hence $a_{11} \equiv a_{22} \equiv \gamma$.

In a similar manner, when $h = 1$, $k \neq 1$, we get $a_{11} \equiv a_{22} \equiv \alpha$.

Next let $h = 1$, $k = 1$, so that

$$R^{-1}P_iR = P_i^a, \quad Q^{-1}P_iQ = P_i^\gamma \quad (i = 1, 2).$$

One of the operations P'_1, P'_2 must be independent of P_1 . As $\gamma^2 \equiv 1 \pmod{p}$, we may assume that P_1 and P'_2 are independent. These will generate I_{p^2} , so that

$$Q^{-1}P_1Q = P_1^{a_{11}}P_2^{a'_{21}}, \quad R^{-1}P'_2R = P_1^{b_{12}}P_2^{b'_{22}}.$$

The abelian conditions from J_Q and J_R are [Eq. (3)]

$$\delta_{12} = b_{12}(a_{11} - \delta) \equiv 0, \quad \delta'_{12} = a_{21}(b_{22} - \alpha) \equiv 0, \quad d_{12} = a_{21}b_{12} \equiv 0.$$

Thus three possibilities arise, viz.,

- (i) $a_{21} \equiv 0, \quad b_{12} \equiv 0 \quad a_{11} \equiv \delta;$
 (ii) $a_{21} \not\equiv 0, \quad b_{12} \equiv 0, \quad b_{22} \equiv \alpha;$
 (iii) $a_{21} \equiv 0, \quad b_{12} \equiv 0.$

For (i), let $P'_1 = P_1^x P_2^{y'}$, $P_2 = P_1^z P_2^{w'}$, whence

$$\begin{aligned} Q^{-1} P'_1 Q &= P_1^{x\gamma} P_2^{y'\gamma} = P_1^{\delta x} P_2^{\delta y'}, \\ R^{-1} P_2 R &= P_1^{z\beta} P_2^{w'\beta} = P_1^{az+b_{12}w} P_2^{b_{22}w'}, \\ (\gamma - \delta)x &\equiv 0, \quad (\gamma - \delta)y \equiv 0, \\ w(b_{22} - \beta) &\equiv 0, \quad z(\alpha - \beta) + b_{12}w \equiv 0. \end{aligned}$$

Hence $\gamma \equiv \delta$ and $k = 1$; but as P_1, P_2 are independent, $w \not\equiv 0$, $b_{22} \equiv \beta$, $\alpha \not\equiv \beta$ and $h \neq 1$, contrary to hypothesis. Since (ii) is likewise excluded, we have $a_{21} \equiv b_{12} \equiv 0$,

$$\begin{aligned} Q^{-1} P_1 Q &= P_1^{\alpha_{11}}, \quad R^{-1} P'_2 R = P_2^{h_{22}}, \\ x(a_{11} - \gamma) &\equiv 0, \quad y(\delta - \gamma) \equiv 0, \\ z(\beta - \alpha) &\equiv 0, \quad w(b_{22} - \beta) \equiv 0, \end{aligned}$$

where $x \not\equiv 0$, $w \not\equiv 0$. Hence when $\alpha \equiv \beta$, $\delta \equiv \gamma$ there results $a_{11} \equiv \gamma$, $b_{22} \equiv \alpha$. We are thus led to a single set of defining relations:

$$\begin{aligned} P_1^p &= P_2^q = Q^r = R^r = 1, \quad P_1 P_2 = P_2 P_1, \quad Q^{-1} P_1 Q = P_1^\gamma, \\ Q^{-1} P_2 Q &= P_2^{\gamma^k}, \quad R^{-1} P_1 R = P_1^\alpha, \quad R^{-1} P_2 R = P_2^{\alpha^h}, \quad RQ = QR, \\ \alpha^r &\equiv 1(p), \quad \gamma^q \equiv 1(p) \quad (h = 1, 2, \dots, r-1; k = 1, 2, \dots, q-1), \end{aligned}$$

or, briefly, say $G = (1 : \gamma 0 : 0 \gamma^k : \alpha 0 : 0 \alpha^h : 1)$. Proceeding to the determination of τ we observe that there are, by hypothesis, two subgroups, $\{P_1\}$, $\{P_2\}$, both permutable with Q and R . In any isomorphism of G with itself either $\{P_1\} \sim \{P_2\}$, $\{P_2\} \sim \{P_1\}$ or else $\{P_1\} \sim \{P_1\}$, $\{P_2\} \sim \{P_2\}$. Hence there are two choices of generators of order p . Every element of G is of the form $\Omega = R^x Q^y P_1^u P_2^v$. Hence $\Omega' = R^{x'} Q^{y'} P_1^{u'} P_2^{v'}$, so that Ω is of order r only when $y \equiv 0 \pmod{q}$ and of order q when $x \equiv 0 \pmod{r}$. Thus the most general operator of order q is $Q_0' = Q^y P_1^u P_2^v$, which transforms G in the same manner as $Q_0 = Q^y$. Similarly $R_0 = R^x$. Employing the new generators $R_0, Q_0, P_{1_0} = P_1, P_{2_0} = P_2$, we get

$$(1 : \gamma 0 : 0 \gamma^k : \alpha 0 : 0 \alpha^h : 1) \sim (1 : \gamma^y 0 : 0 \gamma^{ky} : \alpha^x 0 : 0 \alpha^{hx} : 1).$$

Hence any set of relations involving arbitrary primitive roots (α^a, γ^b) can be transformed into the original set. Next let $P_{1_0} = P_2, P_{2_0} = P_1$. Then

$$(1 : \gamma 0 : 0 \gamma^k : \alpha 0 : 0 \alpha^h : 1) \sim (1 : \gamma^{ky} 0 : 0 \gamma^y : \alpha^{hx} 0 : 0 \alpha^x : 1)$$

if

$$(6) \quad ky \equiv 1 \pmod{q}, \quad hx \equiv 1 \pmod{r}.$$

The group characterized by $[h, k]$ is thus isomorphic with $[x, y]$ when (6) is satisfied. Further τ equals the number of distinct solutions of (6), e. g., when $r = 2$, $\tau = \frac{1}{2}(q + 1)$, and when r is odd, $\tau = \frac{1}{2}(qr + q + r + 1)$.

[3] $i = \text{VI}$, $j = \text{V}$. When $h \neq 1$ we have $Q^{-1}P_jQ = P_{j'}^{a_j}$ ($j = 1, 2$). Assuming that

$$R^{-1}P'_1R = P_1^x P_2^y, \quad R^{-1}P'_2R = P_1^z P_2^w,$$

we derive

$$a_{11}x - z \equiv 0, \quad x - (\iota^p + \iota - a_{11})z \equiv 0,$$

$$a_{22}y - w \equiv 0, \quad y - (\iota^p + \iota - a_{22})w \equiv 0.$$

The elimination of x, y, z, w gives

$$a_{jj}^2 - (\iota^p + \iota)a_{jj} + 1 \equiv 0 \quad (j = 1, 2),$$

whence $a_{jj} = \iota^p$ or ι . Hence a_{11}, a_{22} are galoisian imaginaries* and G , for $i = \text{VI}$, $j = \text{V}$, does not exist.

Before considering the ambiguous case $h = 1$ a few general results must be established.

Let S and T be any set of generators of I_{p^2} , so that $G = \{S, T, Q, R\}$. We may write

$$P'_1 = S^x T^y, \quad P'_2 = S^z T^w,$$

$$Q^{-1}SQ = S^{a_{11}} T^{a_{21}}, \quad Q^{-1}TQ = S^{a_{12}} T^{a_{22}}.$$

Hence

$$Q^{-1}P'_1Q = P'_2 = S^z T^w = S^{a_{11}x + a_{12}y} T^{a_{21}x + a_{22}y},$$

$$Q^{-1}P'_2Q = P_1^{-1}P'_2^{\iota^p + \iota} = S^{-x + (\iota^p + \iota)z} T^{-y + (\iota^p + \iota)w} = S^{a_{11}z + a_{12}w} T^{a_{21}z + a_{22}w},$$

whence results the eliminant

$$D = \begin{vmatrix} x & y & z & w \\ a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ 1 & 0 & a_{11} - \iota & a_{12} \\ 0 & 1 & a_{21} & a_{22} - \iota \end{vmatrix} \equiv 0 \pmod{p},$$

where $t = \iota^p + \iota$. Its expansion gives

$$D_{12}^2 - t(a_{11} + a_{22} - t)D_{12} + a_{22}^2 - a_{11}^2 + t(a_{11} - a_{22}) + 2a_{12}a_{21} + 1 \equiv 0.$$

Now assume $S = P_1$. Then, since $p \equiv -1 \pmod{q}$, $\rho_{Q,p} = 0$ and we may take $Q^{-1}P_1Q \equiv U$ as T . Then

* SERRET, *Cours d'Algebre Supérieure*, cinq. ed. (1885), tome 2, sec. 3, chap. 3. See also DICKSON, *Linear Groups*, pp. 14-19.

$$J_Q = \begin{pmatrix} P_1^{x_1} U^{x_2} \\ P_1^{a_{12}x_2} U^{x_1+a_{22}x_2} \end{pmatrix}, J_Q^q = 1,$$

$$D_{12}^q = \begin{vmatrix} 0 & a_{12} \\ 1 & a_{22} \end{vmatrix}^q \equiv (-a_{12})^q \equiv 1 \pmod{p}.$$

Now $-a_{12}$ cannot be a primitive root of this congruence; for, if so $p \equiv 1 \pmod{q}$, whereas $p \equiv -1 \pmod{q}$ and $q > r$. It follows that $a_{12} \equiv -1 \pmod{p}$ and

$$D \equiv (a_{22} - t)^2 \equiv 0, a_{22} \equiv t \equiv \nu^p + \iota.$$

This gives $I_p = \{P_1 U\}$ and

$$(7) \quad \begin{aligned} Q^{-1}P_1Q &= U, & Q^{-1}UQ &= P_1^{-1}U^{\nu^p+\iota}, \\ R^{-1}P_1R &= P_1^\alpha, & R^{-1}UR &= P_1^\xi U^\eta, \end{aligned}$$

$$\delta_{12} = \begin{vmatrix} -1 & \xi \\ -\nu^p - \iota & \alpha - \eta \end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix} 1 & 0 \\ -\nu^p - \iota & \alpha - \eta \end{vmatrix} \equiv 0,$$

and thus, when $h = 1$, $\eta \equiv \alpha$, $\xi \equiv 0 \pmod{p}$.

Inversely let $P_2 = P_1^{\xi'} U^{\eta'}$. Then

$$R^{-1}P_2R = P_1^{\xi'\alpha} U^{\eta'\alpha} = P_1^{\xi'\alpha} U^{\eta'\alpha}$$

and hence $h = 1$. Thus when $h = 1$ there exists a group

$$G = \{P_1, U, Q, R\} = (1:01:-1\nu^p+\iota:\alpha 0:0\alpha:1),$$

where $\alpha \equiv 1 \pmod{p}$, $p \equiv 1 \pmod{r}$, $\tau = 1$. Also $p \equiv -1 \pmod{q}$ and, in the $GF[p^2]$, $\iota \equiv 1 \pmod{p}$.

[4] $i = V, j = VI$. Since r is necessarily an odd prime, the argument of [3] again gives for G a single type, $G = (1:\gamma 0:0\gamma:01:-1\nu^p+\iota:1)$, with $\gamma \equiv 1 \pmod{p}$, $p \equiv 1 \pmod{q}$, $\tau = 1$. Likewise $p \equiv -1 \pmod{r}$; and $\nu \equiv 1 \pmod{p}$ in the $GF[p^2]$.

[5] $i = VI, j = VI$. Employing as in [3] the theory of the determinant D we are led to the same equations (7), viz.,

$$Q^{-1}P_1Q = U, \quad Q^{-1}UQ = P_1^{-1}U^{\nu^p+\iota_1}, \quad \iota_1^q \equiv 1 \pmod{p}.$$

Let us assume that

$$R^{-1}P_1R = P_2 = P_1^x U^\nu, \quad R^{-1}UR = P_1^z U^w.$$

Then

$$\delta_{12} = \begin{vmatrix} -1 & z \\ -\nu^p - \iota_1 & x - w \end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix} 1 & y \\ -\nu^p - \iota_1 & x - w \end{vmatrix} \equiv 0,$$

$$d_{12} = \begin{vmatrix} -1 & 1 \\ z & y \end{vmatrix} \equiv 0, \quad D_{12} = \begin{vmatrix} x & z \\ y & w \end{vmatrix} \not\equiv 0.$$

Thus

$$z \equiv -y, \quad w \equiv x + (\iota_1^p + \iota_1)y, \quad D_{12} \equiv x^2 + (\iota_1^p + \iota_1)xy + y^2.$$

Since

$$R^{-1}P_2R = P_1^{-1}P_2^{\iota_2^p + \iota_2}, \quad \iota_2' \equiv 1(p),$$

so that

$$R^{-1}U^pR = P_1^{-p^2}U^{xy + (\iota_1^p + \iota_1)y^2} = P_1^{-(x^2+1) + (\iota_2^p + \iota_2)x}U^{-xy + (\iota_2^p + \iota_2)y}.$$

Since P_1 and P_2 are independent, $y \not\equiv 0$; hence

$$(8) \quad 2x + (\iota_1^p + \iota_1)y - (\iota_2^p + \iota_2) \equiv 0,$$

$$(9) \quad y^2 - x^2 + (\iota_2^p + \iota_2)x - 1 \equiv 0.$$

From the latter we at once derive

$$D_{12} = x^2 + (\iota_1^p + \iota_1)xy + y^2 \equiv 1,$$

$$(10) \quad (\iota_2 - \iota_1^2 \iota_2)^2 x^2 - (1 - \iota_1^2)(\iota_2 - \iota_2^3)x + (1 - \iota_1^2 \iota_2^2)(\iota_2^2 - \iota_1^2) \equiv 0,$$

$$(11) \quad (\iota_1 - \iota_1^p)^2 y^2 - (\iota_2 - \iota_2^p)^2 \equiv 0.$$

There always exist integral solutions of (10) and (11), $x = \epsilon_j$, $y = \sigma_j$ ($j = 1, 2$). Thus

$$R^{-1}P_1R = P_1^{\epsilon_1 + (\iota_1^p + \iota_1)\sigma_1}U^{-\sigma_1}, \quad R^{-1}UR = P_1^{\sigma_1}U^{\epsilon_1}.$$

THEOREM. *The two general types of G characterized by the two distinct sets of solutions of (10) and (11), viz. $[\epsilon_1, \sigma_1]$ and $[\epsilon_2, \sigma_2]$ are simply isomorphic.*

In proof, $\sigma_2 \equiv -\sigma_1$, and congruence (8) gives

$$2\epsilon_2 - (\iota_1^p + \iota_1)\sigma_1 - (\iota_2^p + \iota_2) \equiv 0, \quad \epsilon_2 \equiv \epsilon_1 + (\iota_1^p + \iota_1)\sigma_1.$$

Hence the two types of G are characterized by

$$R^{-1}P_1R = P_1^{\epsilon_1 + (\iota_1^p + \iota_1)\sigma_1}U^{-\sigma_1}, \quad R^{-1}UR = P_1^{\sigma_1}U^{\epsilon_1},$$

and

$$R^{-1}P_1R = P_1^{\epsilon_1}U^{\sigma_1}, \quad R^{-1}UR = P_1^{-\sigma_1}U^{\epsilon_1 + (\iota_1^p + \iota_1)\sigma_1}.$$

Let us select a new operation of order q from $\{Q\}$, e. g. $Q' = Q^{-1}$. Then $Q'R = RQ'$, $Q'^{-1}UQ' = P_1$,

$$Q'^{-1}P_1Q' = U^{\tau_1}P_1^{\tau_2} = U^{-1}P_1^{\iota_1^p + \iota_1}, \quad \tau_j = \frac{\iota_1^{(q-j)p} - \iota_1^{q-j}}{\iota_1^p - \iota_1}.$$

The result of selecting Q' and (ϵ_2, σ_2) is thus to interchange P_1 and U and to reproduce the relations given by Q and (ϵ_1, σ_1) . Hence $[\epsilon_2, \sigma_2] \sim [\epsilon_1, \sigma_1]$.

The quantities ι_1 and ι_2 are marks of the $GF[p^2]$ and in that field appertain

respectively to the exponents q and r . Let ρ be any primitive root in the $GF[p^2]$. It is easy to show that $\tau = 1$ and hence we may select*

$$\iota_1 \equiv \rho^{(p^2-1)/q}, \quad \iota_2 \equiv \rho^{(p^2-1)/r},$$

thus

$$G = (1:01:-1, \iota_1^p + \iota_1:\epsilon + (\iota_1^p + \iota_1)\sigma, -\sigma:\sigma\epsilon:1),$$

where

$$\iota_1 \equiv \rho^{(p^2-1)/q}, \quad \iota_2 \equiv \rho^{(p^2-1)/r}, \quad \rho^{p^2-1} \equiv 1; \quad p \equiv -1 \pmod{qr}, \quad \tau = 1,$$

$$(\iota_1 - \iota_1^p)^2 \sigma^2 - (\iota_2 - \iota_2^p)^2 \equiv 0, \quad 2\epsilon + (\iota_1^p + \iota_1)\sigma - (\iota_2^p + \iota_2) \equiv 0.$$

§ 3. The generating function $[k]$.

Consider the relation $R^{-2}P_1R^2 = P_1^{u_1}U^{v_1}$. From it

$$u_{x+1} - (2x + t_1y)u_x + (x^2 + t_1xy + y^2)u_{x-1} \equiv 0,$$

$$u_{x+1} - t_2u_x + u_{x-1} \equiv 0 \quad (t_j = \iota_j^2 + \iota_j; j=1, 2),$$

These recurring formulæ give

$$u_k \equiv [k]_2x - [k-1]_2, \quad v_k \equiv [k]_2y,$$

where

$$[k]_j \equiv \frac{\iota_j^{kp} - \iota_j^k}{\iota_j^p - \iota_j}.$$

Following are some of the properties of the generating function $[k]_j$.

$$(12) \quad \frac{[k+1]_j}{[k]_j} = \frac{1}{t_j} + \frac{1}{t_j} + \frac{1}{t_j} + \dots k \text{ terms},$$

$$(13) \quad [k]_j^2 - [k+1]_j[k-1]_j - 1 \equiv 0,$$

$$(14) \quad [0]_j \equiv 0, \quad [1]_j \equiv 1, \quad [-k]_j \equiv -[k]_j,$$

$$(15) \quad [k+1]_j \equiv [2]_j[k]_j - [k-1]_j,$$

$$(16) \quad \{[k+1]_j - [k-1]_j - [2]_j\} \iota_j^k \equiv (\iota_j^{k+1} - 1)(\iota_j^{k-1} - 1).$$

§ 4. Class (10), $p > q > r$.

We shall consider next groups possessing a single maximal self-conjugate subgroup $H_{p^2q, i}$ of non-abelian type ($i = \text{III, IV, V, VI}$). It is readily shown that class (10, 12), with $i = \text{III}$, must contain an invariant subgroup H_{p^2r} . Class (10) remains to be considered.

[1] $i = \text{IV}$. Here $H_{p^2q, \text{IV}} = \{P, Q\}$ and since $\{P\}$ is self-conjugate in G , $R^{-1}PR = P^\beta$. Since $\rho_{R, q} \equiv 1$ [Eq. (1)], $R^{-1}QR = Q^\gamma$. Hence

$$(QR)^{-1}P(QR) = P^{\alpha\beta} = (RQ^\gamma)^{-1}P(RQ^\gamma) = P^{\beta\alpha\gamma}, \quad \alpha^\gamma \equiv 1 \pmod{p^2},$$

$$\alpha\beta(\alpha^{\gamma-1} - 1) \equiv 0 \pmod{p^2}, \quad \gamma \equiv 1 \pmod{q}.$$

* DICKSON, *Linear Groups*, p. 13.

Hence $\{P_1, P_2, R\}$ is self-conjugate in $\{P_1, P_2, Q, R\} = G$, contrary to hypothesis.

[2] $i = V$. Let $H_{p^2q, V} = \{P'_1, P_2, Q\}$. Assuming that

$$R^{-1}P'_1R = P_1'^{a_{11}}P_2^{a_{21}}, \quad R^{-1}P_2R = P_1'^{a_{12}}P_2^{a_{22}},$$

we deduce

$$a_{11}\alpha(\alpha^{\gamma-1} - 1) \equiv 0, \quad a_{21}(\beta^{\gamma} - \alpha) \equiv 0,$$

$$a_{22}\beta(\beta^{\gamma-1} - 1) \equiv 0, \quad a_{12}(\alpha^{\gamma} - \beta) \equiv 0,$$

where $\alpha^{\gamma} \equiv 1(p)$, $\beta \equiv \alpha^{\lambda}$. Now $\gamma \not\equiv 1 \pmod{q}$. Hence

$$a_{11} \equiv 0, \quad a_{22} \equiv 0, \quad \alpha^{\gamma^{\lambda}} \equiv \alpha, \quad \alpha^{\gamma} \equiv \alpha^{\lambda} \pmod{p},$$

$$\gamma \equiv \lambda \pmod{q}, \quad \alpha^{\gamma^2} \equiv \alpha \pmod{p}, \quad \gamma^2 \equiv 1 \pmod{q}.$$

But γ appertains to the exponent r modulo q , and therefore $r = 2$ and $\gamma \equiv -1 \pmod{q}$. Thus

$$R^{-1}P'_1R = P_2^{a_{21}}, \quad R^{-1}P_2R = P_1'^{a_{12}}, \quad a_{12}a_{21} \equiv 1 \pmod{p}.$$

Then $P_1 = P_1'^{a_{12}}$, P_2, Q, R , generate a group of order $2p^2q$, viz., $G = (1 : \alpha 0 : 0 \alpha^{\gamma-1} : 01 : 10 : -1)$. Also $p \equiv 1(q)$, $\tau = 1$.

[3] $i = VI$. It has been shown [§ 1], that $p \equiv \pm 1 \pmod{r}$.

(a) First let $p \equiv 1(r)$. Then $\rho_{R,p} \geq 2$ and two subgroups $\{P_1\}, \{P_3\}$ may be selected which are permutable with R . If

$$Q^{-1}P_1Q = P_2, \quad Q^{-1}P_2Q = P_1^{-1}P_2^{p+1},$$

then

$$R^{-1}P_1R = P_1^{\beta}, \quad R^{-1}QR = Q^{\gamma}, \quad \gamma \not\equiv 1 \pmod{q}.$$

Since I_p is invariant in G we may assume that

$$P_3 = P_1^x P_2^y, \quad R^{-1}P_2R = P_1^x P_2^y,$$

Hence

$$(QR)^{-1}P_1(QR) = P_1^x P_2^y = (RQ^{\gamma})^{-1}R_1(RQ^{\gamma}) = P_1^{-\beta[\gamma-1]}P_2^{\beta[\gamma]},$$

$$(QR)^{-1}P_2(QR) = P_1^{-\beta+[2]x}P_2^{[2]y} = (RQ^{\gamma})^{-1}P_2(RQ^{\gamma}) = P_1^{-[\gamma-1]x-[\gamma]y}P_2^{[\gamma]x+[\gamma+1]y},$$

$$x \equiv -[\gamma-1]\beta, \quad y \equiv [\gamma]\beta,$$

$$[\gamma]^2 \equiv [\gamma-1]^2 + [2][\gamma-1] + 1,$$

$$[\gamma]\{[\gamma+1] - [\gamma-1] - [2]\} \equiv 0.$$

Now $[\gamma] \not\equiv 0 \pmod{q}$. Since $[-k] \equiv -[k]$ and

$$[\gamma+1] - [\gamma-1] - [2] \equiv (\gamma^{r+1} - 1)(\gamma^{r-1} - 1) \equiv 0 \text{ [Eq. (16)]},$$

there results $\gamma \equiv -1 \pmod{q}$, $\gamma^r \equiv (-1)^r \equiv +1 \pmod{q}$, whence $r = 2$. If $R^{-1}P_3R = P_3^{\alpha}$, then $\alpha \equiv \pm 1 \pmod{p}$,

$$\begin{aligned} w(y \mp 1) &\equiv 0, & xw + z(\beta \mp 1) &\equiv 0, \\ w(-\beta \mp 1) &\equiv 0, & [2]\beta w + z(\beta \mp 1) &\equiv 0. \end{aligned}$$

First let the upper sign hold. If $\beta \equiv 1$, then $w \equiv 0$ which is impossible, since P_1, P_3 are independent. Hence $\beta \equiv -1$, $x \equiv -[2]$, $y \equiv +[1] \equiv +1$. Likewise if we use the lower sign, $\beta \equiv +1$, $x \equiv +[2]$, $y \equiv -[1] \equiv -1$. We thus obtain the two sets of defining relations:

$$(1:01:-1\iota^p + \iota:\mp 10:\iota^{\mp p} + \iota^{\mp 1}, \pm 1:-1).$$

To determine τ , let $Q_0 = Q^x$, $R_0 = R$, $P_{1_0} = P_1$, $P_{2_0} = P_1^{-[x-1]}P_2^{[x]}$; there results

$$\{P_{1_0}, P_{2_0}, Q_0, R_0\} = (1:01:-1\iota^{xp} + \iota^x:\mp 10:\pm[x-1]\mp[2][x], \pm[x]:-1).$$

But

$$\pm[x-1]\mp[2][x] \equiv \mp[x+1] \equiv \mp(\iota^{xp} + \iota^x)\mp[x-1],$$

[Eq. (15)]. Hence

$$\{P_{1_0}, P_{2_0}, Q_0, R_0\} = (1:01:-1\iota^{xp} + \iota^x:\mp 10:\mp(\iota^{xp} + \iota^x), \pm 1:-1) \sim G.$$

Thus the same defining relations are reproduced with ι replaced by ι^x , and so $\tau = 1$.

It will now be proved that these two types are simply isomorphic. Select new operators as follows:

$$q_1 = Q, \quad r_1 = R, \quad p_1 = P_1^a P_2^b, \quad p_2 = P_1^{-b} P_2^{a+[2]b} = q_1^{-1} p_1 q_1.$$

Then using the first set of defining relations we will have

$$\text{if } q_1^{-1} p_2 q_1 = p_1^{-1} p_2^{\iota^p + \iota}, \quad r_1^{-1} p_1 r_1 = p_1, \quad r_1^{-1} p_2 r_1 = p_1^{\iota^p + \iota} p_2^{-1}, \quad r_1^{-1} q_1 r_1 = q_1^{-1}$$

$$2a + [2]b \equiv 0 \pmod{p}.$$

Hence when a new operator $p_1 = P_1^a P_2^b$ is selected, where a and b are solutions of $2a + (\iota^p + \iota)b \equiv 0 \pmod{p}$, the first type is transformed into the second. They are therefore isomorphic.

(b) When $p \equiv -1(r)$, r odd, $\rho_{R,p} = 0$. As before, we deduce

$$Q^{-1}P_1Q = P_2, \quad Q^{-1}P_2Q = P_1^{-1}P_2^{\iota_1^p + \iota_1}, \quad \iota_1^q \equiv 1(p),$$

$$R^{-1}P_1R = P_3, \quad R^{-1}P_3R = P_1^{-1}P_3^{\iota_2^p + \iota_2}, \quad \iota_2^r \equiv 1(p),$$

Let $P_3 = P_1^x P_2^y$ and $R^{-1}P_2R = P_4 = P_1^z P_2^u$. Then

$$(17) \quad R^{-1}P_2^yR = P_1^{-(x^2+1)+[2]_2x}P_2^{y[2]_2-y^2} = P_1^{-[y-1]_2xy-[y]_2y^2}P^{[y]_2xy+[y+1]_2y^2}.$$

In addition to the latter, but not independent of them, we have the congruences derived from

$$(18) \quad (QR)^{-1}P_2^r(QR) = (RQ^r)^{-1}P_2^r(RQ^r).$$

The equations (17) and (18) give us the dialytic eliminant

$$\Delta_{12} = \{\iota_2^p + \iota_2\} \{[\gamma]_1^2 - (\iota_2^{2p} + \iota_2^2)[\gamma]_1 + 1\} \{(\iota_1^{r+1} - 1)(\iota_1^{r-1} - 1)\}^2 \equiv 0.$$

Now $[\gamma]_1$ is an integer, and since $r \neq 2$, and $\gamma \neq -1$, it follows that $\gamma \equiv 1 \pmod{q}$, contrary to hypothesis. Hence when $p \equiv -1 \pmod{r}$ and r is odd, no corresponding group G exists.

The results of this section may be summarized in the following

THEOREM. *A group G_{p^2qr} ($p > q > r$) always contains a maximal self-conjugate subgroup H of order p^2q . If H is the only maximal invariant subgroup of G and if r is odd, then $N_q = 1$ and H is necessarily abelian. If r is even ($r = 2$) and $p \equiv 1 \pmod{q}$ there exists one type whose subgroup H_{p^2q} is non-abelian, and if r is even and $p \equiv -1 \pmod{q}$ there exists a second type possessing a non-abelian H_{p^2q} . These two types of G contain respectively q and pq operators (and subgroups) of order 2, and in each type $N_q = p^2$. Moreover, with exception of the two types just described, every group of order p^2qr ($p > q > r$), in which $N_r \equiv 0 \pmod{q}$, possesses an abelian maximal self-conjugate subgroup H_{p^2q} .*

A general summary of all the existent types of G follows. Except for ι and ρ , every parameter occurring in the tables is an integer; while ι and ρ are marks of the $GF[p^2]$. See footnote on the second page of the paper.

TABLE 1. $p > q > r$.

I_p non-cyclical; $P_1^p = P_2^p = Q^q = R^r = 1, P_1P_2 = P_2P_1, I_p$ cyclical; $P_1^p = Q^q = R^r = 1$.
Case (a), $QR = RQ$.

Class	$Q^{-1}P_1Q$	$Q^{-1}P_2Q$	$E^{-1}P_1R$	$E^{-1}P_2R$	Parameters.	Arith. Rel.	τ
[12...12]	P_1	P_1	P_1	P_1	.	.	1
"	P_1	P_2	P_1	P_2	.	.	1
[3467891012]	P_1^a	P_2	P_1	P_2	$\alpha^q \equiv 1(p)$	$p \equiv 1(q)$	1
[3489101112]	P_1	P_2	P_1	P_2	$\alpha^r \equiv 1(p)$	$p \equiv 1(r)$	1
[46891012]	P_1^a	P_2	P_1	P_2	$\alpha^q \equiv \beta^r \equiv 1(p)$	$p \equiv 1(qr)$	1
[78910]	P_1^a	.	P_1	.	$\alpha^q \equiv 1(p^2)$	$p \equiv 1(q)$	1
"	P_1^a	P_2^a	P_1	P_2	$\alpha^q \equiv 1(p)$	$p \equiv 1(q)$	$\frac{1}{2}(q+1)$
"	P_2	$P_1^{-1}P_2^{q+1}$	P_1	P_2	$i^q \equiv 1(p)$	$p \equiv -1(q)$	1
[891012]	P_1^a	P_2	P_1	P_2^8	$\alpha^q \equiv \beta^r \equiv 1(p)$	$p \equiv 1(qr)$	1
[9101112]	P_1	.	P_1	P_1^a	$\alpha^q \equiv 1(p^2)$	$p \equiv 1(r)$	1
"	P_1	P_2	P_1^a	P_2^a	$\alpha^r \equiv 1(p)$	$p \equiv 1(r)$	1 or $\frac{1}{2}(r+1)$
"	P_1	P_2	P_2	$P_1^{-1}P_2^{q+1}$	$i^r \equiv 1(p)$	$p \equiv -1(r)$	1
[8910]	P_1^8	P_2^8	P_1^a	P_2	$\alpha^r \equiv \beta^q \equiv 1(p)$	$p \equiv 1(qr)$	$q-1$
[91012]	P_1^a	P_2	P_1^8	P_2^8	$\alpha^q \equiv \beta^r \equiv 1(p)$	$p \equiv 1(qr)$	$r-1$
[910]	P_1^a	.	P_1^8	.	$\alpha^q \equiv \beta^r \equiv 1(p^2)$	$p \equiv 1(qr)$	1
"	P_1	P_2^*	P_1^a	P_2^a	$\gamma^q \equiv \alpha^r \equiv 1(p)$	$p \equiv 1(qr)$	$\frac{1}{2}(q+1)$ or $\frac{1}{2}(r+1)(q+1)$
"	P_2	$P_1^{-1}P_2^{q+1}$	P_1^a	P_2^a	$i^q \equiv 1(p)$	$p \equiv -1(q)$	1
"					$\alpha^r \equiv 1(p)$	$p \equiv 1(r)$	
"	P_2	$P_1^{-1}P_2^{q+1}$	P_1^a	P_2^a	$\rho = \text{prim. root in } GF[p^2]; i_1, i_2 = \rho^{(p^2-1)/q, r}$	$p \equiv -1(qr)$	1
"	P_2	$P_1^{-1}P_2^{\frac{p^2-1}{q} + \rho^q}$	P_1^a	$P_2^{-\sigma}$	$2\epsilon + [2]_1\sigma - [2]_2 \equiv 0$		
"					$(i_1 - i_1^r)^2\sigma^2 - (i_2 - i_2^r)^2 \equiv 0$		
"	P_1	P_2	P_2	$P_1^{-1}P_2^{q+1}$	$i^r \equiv 1(p)$	$p \equiv -1(r)$	1
"					$\gamma^q \equiv 1(p)$	$p \equiv 1(q)$	

Case (b). $R^{-1}QR = Q^{\gamma^A}$; $\gamma \equiv 1(q)$.

Class.	$Q^{-1}P_1Q$	$Q^{-1}P_2Q$	$R^{-1}P_1R$	$R^{-1}P_2R$	Parameters.	Arith. rel.	τ
[256101112]	P_1	P_2	P_1	P_2	$h = 1$	$q \equiv 1(r)$	1
"	P_1	.	P_1	.	$h = 1$	$q \equiv 1(r)$	1
[56101112]	P_1	P_2	P_1^a	P_2	$h = 1, 2 \dots r-1$ $\alpha^r \equiv 1(p)$	$p \equiv q \equiv 1(r)$	$r-1$
[101112]	P_1	.	P_1^a	.	$h = 1, 2 \dots r-1$ $\alpha^r \equiv 1(p^2)$	$p \equiv q \equiv 1(r)$	$r-1$
"	P_1	P_2	P_1^a	P_2^k	$h, k=1, 2 \dots r-1$ $\alpha^r \equiv 1(p)$	$p \equiv q \equiv 1(r)$	1 or $\frac{1}{2}(r^2-1)$
"	P_1	P_2	P_2	$P_1^{-1}P_2^{\gamma^+}$	$h = 1, 2 \dots r-1$ $\epsilon^r \equiv 1(p)$	$p \equiv -q \equiv -1(r)$	$r-1$
[10]	P_1^a	$P_2^{\alpha^r-1}$	P_2	P_1	$h = 1, \gamma \equiv -1$ $\alpha^r \equiv 1(p)$	$r = 2$ $p \equiv 1(q)$	1
"	P_2	$P_1^{-1}P_2^{\gamma^+}$	P_1^{-1}	$P_1^{\gamma^+}P_2^{-1}$	$h = 1, \gamma \equiv -1$ $\epsilon^r \equiv 1(p)$	$r = 2$ $p \equiv -1(q)$	1

TABLE 2. $q > p > r$.

I_p , non-cyclical; $P_i^r = Q^r = R^r = 1$ ($i = 1, 2$), $P_1P_2 = P_2P_1$, $RP_2 = P_2R$,
 I_r , cyclical; $P_1^r = Q^r = R^r = 1$, $RP_1 = P_1R$.

Class.	$P_1^{-1}QP_1$	$P_2^{-1}QP_2$	$R^{-1}QR$	$R^{-1}P_1R$	Parameters.	Arith. Rel.	τ
[1234561112]	Q^a	.	Q	P_1	$\alpha^p \equiv 1(q)$	$q \equiv 1(p)$	1
[12511]	Q^a	.	Q	P_1	$\alpha^{p^2} \equiv 1(q)$	$q \equiv 1(p^2)$	1
[1112]	Q^a	.	Q^r	P_1	$\alpha^p \equiv \gamma^r \equiv 1(q)$	$q \equiv 1(pr)$	1
[11]	Q^a	.	Q^r	P_1	$\alpha^{p^2} \equiv \gamma^r \equiv 1(q)$	$q \equiv 1(p^2r)$	1
[1251112]	Q	Q^a	Q	P_1	$\alpha^p \equiv 1(q)$	$q \equiv 1(p)$	1
[251112]	Q	Q^r	Q^a	P_1	$\gamma^p \equiv \alpha^r \equiv 1(q)$	$q \equiv 1(pr)$	1
[4561112]	Q	Q^a	Q	P_1^s	$\alpha^p \equiv 1(q)$ $\delta^r \equiv 1(p)$	$q \equiv 1(p)$ $p \equiv 1(r)$	1
[561112]	Q	Q^a	Q^{γ^A}	P_1^s	$\alpha^p \equiv \gamma^r \equiv 1(q)$ $\delta^r \equiv 1(p)$	$q \equiv 1(pr)$ $p \equiv 1(r)$	$r-1$

TABLE 3. $q > r > p$.*Case (a).*

I_p , non-cyclical; $P_i^p = Q^q = R^r = 1$ ($i = 1, 2$), $P_1P_2 = P_2P_1$, $RQ = QR$,
 I_p , cyclical; $P_1^{p^2} = Q^q = R^r = 1$, $QR = RQ$.

Class.	$P_1^{-1}QP_1$	$P_2^{-1}QP_2$	$P_1^{-1}RP_1$	$P_2^{-1}RP_2$	Parameters.	Arith. Rel.	τ
[12345678]	Q	.	R^a	.	$\alpha^p \equiv 1(r)$	$r \equiv 1(p)$	1
[1237]	Q	.	R^a	.	$\alpha^{p^2} \equiv 1(r)$	$r \equiv 1(p^2)$	1
[123456]	Q^a	.	R^{β^a}	.	$\alpha^p \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv r \equiv 1(p)$	$p - 1$
[125]	Q^a	.	R^{β^a}	.	$\alpha^{p^2} \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv 1(p^2)$ $r \equiv 1(p)$	$p - 1$
[234]	Q^{a^2}	.	R^β	.	$\alpha^p \equiv 1(q)$ $\beta^{p^2} \equiv 1(r)$	$r \equiv 1(p^2)$ $q \equiv 1(p)$	$p - 1$
[12]	Q^a	.	R^{β^a}	.	$\alpha^{p^2} \equiv 1(q)$ $\beta^{p^2} \equiv 1(r)$	$q \equiv r \equiv 1(p^2)$	$p^2 - 1$
[12345678]	Q	Q	R	R^a	$\alpha^p \equiv 1(r)$	$r \equiv 1(p)$	1
[123]	Q	Q^a	R	R^{β^a}	$\alpha^p \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv r \equiv 1(p)$	$p - 1$
[1235]	Q	Q^β	R^a	R	$\alpha^p \equiv 1(r)$ $\beta^p \equiv 1(q)$	$q \equiv r \equiv 1(p)$	1

Case (b). The simple group G_{161} , $p = 2$, $q = 5$, $r = 3$.

$$Q^5 = 1, \quad P^2 = 1, \quad (QP)^3 = 1, \quad [R = QP].$$